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TITLE- A Birth-Death Process Associated With A Redundant Repairable System

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ABSTRACT

A finite birth-death process $\eta(t)$, $t > 0$, with density dependent birth-death rates and reflecting barriers is a model for a redundant repairable system consisting of N identical computers and repair crews, where $\eta(t)$ is the number of computers in operation at time t. The transition probabilities of this process are calculated together with its asymptotic distribution. The transition probabilities are then used to derive a formula for the Laplace transform of the distribution of the first time to system failure.

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FROM: G. R. Andersen

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TECHNICAL MEMORANDUM

INTRODUCTION

We will consider a system composed of N computers together with one repair crew assigned to each computer. It is supposed that all N computers are in operation at time $t = 0$ and that the failure times of the N computers are independent, each distributed according to an exponential law with mean $1/\lambda$, $\lambda > 0$. If a computer fails it is transferred to a state of repair. It is assumed that the repair times of the N computers are independent and that each is exponentially distributed with mean $1/\mu$, $\mu > 0$. When a failed computer is repaired it is returned to the operational state. The detection of any change in state (operational or nonoperational) is assumed perfect and the switching between these states is assumed to be instantaneous.

We will say that this system is in the state i at time t, if exactly i computers are in operation at time t, $(i=0, 1, \ldots, N)$. If we suppose that each of the N computers performs the same task, a system failure occurs (for the first time) at the first instant in which the system is in the state zero. Since we do not suppose that the zero state is absorbing,* the system returns to the state 1 after a random waiting time** and then continues its random evolution in time as if it has started from the state $i = 1$ at time zero.

Thus, if $\eta(t)$ denotes the state of the system at time t, it is the object of this memorandum to determine the probability distribution of $\eta(t)$ for each $t > 0$ together with the transition probabilities of the process. These are given in section 2 along with the asymptotic (stationary) distribution of the states of the system.

This asymptotic distribution was used by J. J. Rocchio [4] as part of an extensive reliability study concerning the above described system.

---

*This case was treated by I. D. Nehama [3].

**The waiting time is exponential with parameter $1/N\mu$.

The remainder of the memorandum is either directly or indirectly concerned with finding the probability distribution of the first passage of the system from the state N to the state i (i < N). Some explicit formulas are given for the Laplace transform of the first passage from N to zero state and from N to the state 3.

After this memo was written, the results of S. Karlin and J. L. McGregor were found in the book by R. E. Barlow and E. Proschan [5]. Karlin and McGregor [6] give an integral representation of the transition probabilities of birth-death processes which involve a sequence of polynomials and a positive regular measure on $(0, \infty)$. In the problem under consideration here the polynomials are the Krawtchouk polynomials; the measure is induced by the binomial distribution and the results correspond exactly to those given in section 2 (equations (2.7) and (2.9)) of this memo [5; p. 145]. The methods are, however, entirely different--the main tool used here being the well-known probability generating function.

1. The Transition Probabilities

Let $P_{ij}(t)$ denote the probability that the system passes from the state i to the state j during a time interval of length t; that is,

$$(1.1)\quad P_{ij}(t) = P\{\eta(t+\tau) = j \mid \eta(\tau) = i\}^{*},\ i,j=0,1,\ldots,N;\ t,\tau \geqq 0.$$

Then the system described in the introduction implies that

$$(1.2)\quad \begin{cases} P_{ii+1}(h) = (N-i)\mu h+o(h),\ i=0,1,\ldots,N-1 \\ P_{ii-1}(h) = i\lambda h + o(h),\ i=1,2,\ldots,N \\ P_{i,i}(h) = 1 - (i\lambda + (N-i)\mu)\, h + o(h),\ i=0,1,2,\ldots,N, \\ P_{ij}(h) = o(h) \text{ if } j \neq i-1, \text{ or } i, \text{ or } i+1, \text{ as } h \to 0 \text{ and} \\ P_{ij}(0) = \begin{cases} 1, \text{ if } i=j \\ 0, \text{ if } i\neq j \end{cases} . \end{cases}$$

---

*The stationarity of the transition probabilities is a consequence of the fact that the failures and repair distribution are exponential.

For completeness, we set

$$P_{N,N+1}(h) = 0 \text{ and } P_{0,-1}(h) = 0 \text{ for all } h \geqq 0.$$

The stochastic process $\{\eta(t): t>0\}$ is therefore a finite birth-death process with reflecting barriers at the states 0 and N and density dependent birth and death rates. It is a particular case of a continuous time parameter, finite state Markov process with stationary transition probabilities and, as such, most of the basic mathematical facts concerning this process can be found in the book of Kai Lai Chung [1].

Using the system of infinitesimal transition probabilities given in (1.2), it is well known that, for each i (i=0,1,...,N), one can obtain the following set of N+1 differential equations:

$$P'_{i0}(t) = -N\mu P_{i0}(t) + \lambda P_{i1}(t),$$

$$P'_{iN}(t) = -N\lambda P_{iN}(t) + \mu P_{iN-1}(t), \text{ and}$$

(1.3)
$$P'_{iK}(t) = -(K\lambda+(N-K)\mu)P_{iK}(t) + \mu(N-K+1)P_{iK-1}(t) + (K+1)\lambda P_{iK+1}(t)$$

$$\text{for } K = 1,2,\ldots,N-1;\ t>0 .$$

The solution of (1.3), for each i=0,1,...,N, is obtained in section 2 using $P_{ij}(0) = \delta_{ij}$. A basic assumption stated in the introduction required that all N computers were in operation at the time t=0. Hence, the distribution of $\eta(t)$ is given by

(1.4) $$P(\eta(t)=K) = P_{NK}(t),\ K=0,1,\ldots,N;\ t \geqq 0 .$$

2. <u>The Solution of (1.3) Using Probability Generating Functions</u>

For each i, i=0,1,...,N let M_i be the probability generating function of $\{P_{iK}(t): K=0,1,\ldots,N\}$. Then

$$(2.1) \quad M_i(s,t) = \sum_{K=0}^{N} S^K P_{iK}(t), \ t \geq 0, \ |S| \leq 1.$$

By differentiating M_i with respect to t and using (1.3) we obtain the following partial differential equation:

$$(2.2) \quad \frac{\partial M_i(S,t)}{\partial t} + (\mu S+\lambda)(S-1) \frac{\partial M_i(S,t)}{\partial S} = \mu N(S-1) M_i(S,t) \ ,$$

subject to the initial conditions

$$(2.3) \quad M_i(S,0) = S^i$$

for each i, $i=0,1,\ldots,N$.

For each i, the solution* is given by

$$(2.4) \quad M_i(S,t) = (q_0(t)+p_0(t)S)^i(q_1(t)+p_1(t)S)^{N-i} \ ,$$

where

$$(2.5) \quad \begin{cases} p_0(t) = \alpha + \beta e^{-(\lambda+\mu)t}, \ p_1(t) = \alpha(1-e^{-(\lambda+\mu)t}) \\ q_0(t) = 1 - p_0(t), \ q_1(t) = 1 - p_1(t) \end{cases}$$

and

$$(2.6) \quad \alpha = \frac{\mu}{\mu+\lambda}, \quad \beta = \frac{\lambda}{\mu+\lambda} \ .$$

---

*The derivation is given in Appendix I.

It follows immediately that $P_{iK}(t)$ is the coefficient of s^K in equation (2.4). Before we write down the general result, let us note that when i=N, the right-hand side of (2.4) is just

$$\sum_{K=0}^{N} \binom{N}{K} p_0^K(t)\, q_0^{N-K}(t)\, s^K .$$

Hence, from (1.4) we obtain the distribution $\eta(t)$:

$$(2.7)\quad P\{\eta(t)=K\} = \binom{N}{K} p_0^K(t)\, q_0^{N-K}(t)$$

$$= \binom{N}{K} \frac{\lambda^{N-K}}{(\mu+\lambda)^N} \left(\mu+\lambda e^{-(\lambda+\mu)t}\right)^K \left(1-e^{-(\lambda+\mu)t}\right)^{N-K}$$

K=0, 1,..., N.

Therefore, the distribution of $\eta(t)$ is binomial with parameters N and $p_0(t)$ for each fixed $t>0$. It is clear that the process $\eta(t)$ is nonstationary; however, passing to the limit in (2.7) we find that

$$\lim_{t\to+\infty} P\{\eta(t)=K\} = p_k\ , \quad (k=0,1,\ldots,N)\quad ,$$

where

$$(2.8)\qquad p_k = \binom{N}{k} \frac{\mu^k \lambda^{N-k}}{(\mu+\lambda)^N} \quad .$$

The finite sequence $\{p_0, p_1, \ldots, p_N\}$ then represents the stationary asymptotic distribution of the system referred to in the introduction. The significance of the numbers p_k will be made clear in the next section.

Returning to equation (2.4) we easily find that

$$(2.9)\quad P_{ik}(t) = \sum_{r=0}^{k} \binom{N-i}{k-r}\binom{i}{r} p_0^r(t) q_0^{i-r}(t) p_1^{k-r}(t) q_1^{N-i-k+r}(t)$$

where, as usual, the binomial coefficients are set equal to zero when the lower factor is less than zero or greater than the upper factor.

The complete probabilistic structure of the process $\eta(t)$, $t > 0$, is now known, in that the joint distribution,

$$P\,\{\eta(t_1) = i_1, \ldots, \eta(t_\eta) = i_\eta\}\ ,$$

can be calculated explicitly for any time points $t_1 < t_2 < \ldots < t_\eta$ and states $i_1, \ldots, i_\eta$ $(\eta = 1,2,\ldots)$ by using equations (2.7), (2.9) and the Markovian property.

3. A Related Process

Let $C_k(t)$ be the stochastic process obtained from $\eta(t)$ by setting

$$(3.1)\quad C_k(t) = \begin{cases} 1, & \text{if } \eta(t) = k \\ 0, & \text{if } \eta(t) \neq k \end{cases} .$$

Then it is easily shown that the integral

$$(3.2)\qquad Z_k(T) = \int_0^T C_k(t)dt,\ T > 0$$

exists in the quadratic mean and so defines another stochastic process $\{Z_k(t):\ t > 0\}$.

In an obvious sense, $Z_k(t)$ represents the length of time that the process $\eta(t)$ is in the state k, if the system has operating for T units of time. That is, for a particular realization of the process $\eta(t)$, $Z_k(T)$ is the sum of the lengths of those subintervals of (0,T) during which the system is in the state k.

Now, using (3.2), it is clear that

$$EZ_k(T) = \int_0^T P\{\eta(t)=k\}dt \quad . \tag{3.3}$$

Hence, since $P\{\eta(t)=k\} \to p_k$, we have that

$$\frac{EZ_k(T)}{T} \to p_k \quad (T\to+\infty) \quad . \tag{3.4}$$

In fact, it can be shown [3] that $Z_k(T)/T$ converges stochastically to be constant p_k. Thus, since $EZ_k(T)$ is the mean length of time* that the system is in the state k, during the time interval (0,T), p_k $(k=0,1,\ldots,N)$ should be interpreted as the average mean length of time that the system will spend in the state k, if the system operates for a "long time".

It is conceivable then that the quantity $EZ_k(T)/T$ might be of some use in describing the availability of the system for relatively short time periods of length T. For example, if k=0, then from equations (2.7) and (3.3), we obtain

---

*The "mean length of time" does not refer to a time average.

$$\frac{EZ_0(T)}{T} = \frac{1}{T}\int_0^T q_0^N(t)dt =$$

(3.5)

$$= \left(\frac{\lambda}{\mu+\lambda}\right)^N \left[1+\frac{1}{T}\sum_{j=0}^{N-1} \binom{N}{j}(-1)^{N-j}\,\frac{(1-e^{-(\lambda+\mu)(N-j)T}}{(\mu+\lambda)(N-j)}\right]$$

as the average mean length of time that the system will be inoperative during an interval of length T if the system starts from the state N. This quantity is considerably less than p_0 for small values of T; in fact, it is less than $q_0^N(T)$, the probability of being in the zero state at time T and $q_0^N(T) < p_0$.

4. First Passage Times

We now define a random variable α_{Ni}; whose value is the time of first passage from the state N to the state i. That is,

(4.1) $$\alpha_{Ni} = \inf\{t: \ \eta(t)=i, \ \eta(0) = N\} \ , \ 0 \leqq i < N \ .$$

Let F_{Ni} be the d.f. of this random variable. Then a simple heuristic argument* leads to the relation:

(4.2) $$P_{Ni}(t) = \int_0^t P_{ii}(t-s) \ dF_{Ni}(s) \ ,$$

where P_{Ni} and P_{ii} are given by equation (2.9).

---

*For the proof see [1], page 196 or page 205.

Hence, taking the Laplace transforms of both sides of (4.2), using the convolution theorem and the fact that F'_{Ni} exists, we obtain

$$L(F'_{Ni})(s) = \frac{L(P_{Ni})(s)}{L(P_{ii})(s)} \quad , s > 0. \tag{4.3}$$

For example, for i=0 it is easily seen* that

$$L(F'_{N0})(s) \tag{4.4}$$

$$= \frac{N!\lambda^N}{(-\mu)^N N! + \sum_{K=1}^{N} \binom{N}{k}(-\mu)^{N-k}(N-k)!(s+NA)(s+(N-1)A)\ldots(s+[N-k+1]A)}$$

where $A = \mu+\lambda$.**

Formal inversion of (4.4) yields the density function of the first passage to zero from state N. Computationally, this should not be difficult for fixed values of N, λ and μ. But if there is a closed form solution, I have not found it.

However, as is well-known, certain results can be obtained from the Laplace transform. For example, the mean-time-to failure m_{N0}, from the state N to the state 0 is just the Laplace transform of $1-F_{N0}$ evaluated at the origin. That is, since

$$\frac{1-sL(F_{N0})(s)}{s} = \int_0^\infty e^{-st}\,(1-F_{N0}(t))dt$$

and

$$sL(F_{N0}) = L(F'_{N0})$$

---

*For the proof see Appendix II.

**This result is also contained in [3] although the methods used are quite different.

we find that

$$(4.7)\quad m_{N0} = \frac{(-\mu)^{N-1}N!+A\sum_{k=2}^{N} \binom{N}{k}(N-k)!(-\mu)^{N-k}k(N-\frac{k-1}{2})}{(-\mu)^{N}N!+ \sum_{k=1}^{N} \binom{N}{k}(-\mu)^{N-k}(N-k)!A^{k}N(N-1)\ldots(N-k+1)}$$

where $A = \mu+\lambda$.

An interest has been expressed in the d.f., F_{N3}, of the first passage time from the state N to the state 3.* We can proceed as before with i=3. However, the computations are very much more involved and would have to be carried out in specific cases. In an effort to minimize these difficulties, at least for small values of N (e.g., N=5) we introduce the "first exit time from the state N":

$$\rho_N = \inf\{t: \ t > 0, \ \eta(t) \neq N\} .$$

It is well known that ρ_N is exponentially distributed with mean $1/N\lambda$. It then follows (c.f., Chung [1]) that we can write

$$(4.8)\qquad F_{N3}(t) = \int_0^t (1-e^{-\lambda N(t-y)}) \, dR_{N3}(y) ,$$

where, in our case, R_{N3} is just the d.f. of the first passage time from the state N-1 to the state 3 and so satisfies

$$(4.9)\qquad P_{N-1\ 3} = P_{33} * R_{N3} .$$

*This question was raised by J. J. Rocchio and was motivated by changes introduced in the basic model.

Hence, in terms of Laplace transforms

$$L(F_{N3})(s) = \frac{N\lambda}{s(s+N\lambda)} L(R'_{N3})(s)$$

where $L(R'_{N3})$ satisfies

$$L(P_{N-1\ 3})(s) = L(P_{33})(s)\ L(R'_{N3})(s)\ .$$

Some computations were made with N = 5, but are not included here because of the cumbersome nature of the formula. If there is any interest in this formula, it may be obtained from the author.

G. R. Andersen

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Attachments
  References
  Appendices I - II

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APPENDIX I

Consider the partial differential equation (2.2):

$$\text{(i)} \qquad \frac{\partial M_i(s,t)}{\partial t} + (\mu s+\lambda)(s-1)\frac{\partial M_i(s,t)}{\partial s} = \mu N(s-1)M_i(s,t)$$

where

$$\text{(ii)} \qquad M_i(s,0) = s^i$$

for each i, $i=0,1,\ldots,N$; $|s| < 1$.

The auxiliary equation

$$\frac{ds}{(\mu s+\lambda)(s-1)} = dt = \frac{dM_i}{\mu N(s-1)M_i}$$

yields the following two solutions:

$$\frac{\mu s+\lambda}{s-1} e^{(\lambda+\mu)t} = \text{constant}$$

and

$$M_i(s,t)(\mu s+\lambda)^{-N} = \text{constant} .$$

The general solution is then known to be of the form

$$\text{(iii)} \qquad M_i(s,t)(\mu s+\lambda)^{-N} = f\left(\frac{\mu s+\lambda}{s-1} e^{(\lambda+\mu)t}\right)$$

for some function f.

Using (ii) we see that

$$f\left(\frac{\mu s+\lambda}{s-1}\right) = \frac{s^{i}}{(\mu s+\lambda)^{N}}$$

for $|s| < 1$. Setting $\xi = (\mu s+\lambda)/(s-1)$ we find that

(iv) $$f(\xi) = \frac{(\xi+\lambda)^{i}(\xi-\mu)^{N-i}}{(\xi(\mu+\lambda))^{N}} .$$

Combining (iii) and (iv) we obtain

$$M_{i}(s,t) = \frac{[\mu s+\lambda+(\lambda s-\lambda)e^{-(\mu+\lambda)t}]^{i}[\mu s+\lambda-(\mu s-\lambda)e^{-(\mu+\lambda)t}]^{N-i}}{[\mu+\lambda]^{N}}$$

This is equivalent to equation (2.4).

APPENDIX II

From equation (4.3) we have that

(i) $$L(F'_{N0})(s) = \frac{L(P_{N0})(s)}{L(P_{00})(s)} \quad , s > 0 .$$

Using (2.7) we have

$$L(P_{N0})(s) = \int_0^\infty e^{-st} P_{N0}(t)dt$$

$$= \beta^N \int_0^\infty e^{-st}(1-e^{-At})^N dt$$

$$= \frac{\beta^N}{A} \int_0^1 y^{\frac{s}{A}-1} (1-y)^N dy$$

where $A = \mu+\lambda$ and $\beta = \lambda/(\mu+\lambda)$.

Since the right-hand side of the last equation is a Beta function we find that

(ii) $$L(P_{N0})(s) = \frac{\beta^N}{A} \frac{\Gamma(\frac{s}{A})\Gamma(N+1)}{\Gamma(\frac{s}{A}+N+1)} = \frac{\beta^N N! A^N}{s(s+A)\ldots(s+NA)}$$

Similarly, from equation (2.9) with i=0, k=0 we obtain

$$L(P_{00})(s) = \int_0^\infty e^{-st}(\beta+\alpha e^{-At})^N dt$$

where $A = \mu+\lambda$, $\beta = 1-\alpha$.

It follows that

$$\text{(iii)} \qquad L(P_{00})(s) = \sum_{k=0}^{N} \binom{N}{k} \frac{(-\mu)^{N-k}\ (N-k)!}{s(s+A)\ldots(s+(N-k)A)} .$$

Combining (i), (ii) and (iii) we obtain

$$L(F'_{N0})(s) = \frac{N!\ \lambda^N}{(-\mu)^N N! + \sum_{k=1}^{N} \binom{N}{k} (-\mu)^{N-k} (s+[N-k+1]A)\ldots(S+NA)(N-K)!}$$

As an example, we note that for N = 2

$$F'_{N0}(t) = \frac{4\lambda^2}{\sqrt{\mu^2+6\lambda\mu+\lambda^2}}\, e^{-\frac{1}{2}(\mu+\lambda)t} \sinh\{\tfrac{1}{2} \sqrt{\mu^2+6\mu\lambda+\lambda^2}\, t\}$$

if $t \geq 0$.